380

NO

National Aeronautics and Space Administration

HOUSTON, TEXAS

Manned Spacecraft Center

NASA

MSC-IN-64-ED-7

MSC INTERNAL TECHNICAL NOTE

A MODIFICATION OF HERRICK'S SOLUTION OF THE TWO-BODY PROBLEM FOR ALL CASES

BY

William H. Goodyear
Lawrence F. Guseman, Jr.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

MANNED SPACECRAFT CENTER

HOUSTON, TEXAS

July 28, 1964

MSC-IN-64-ED-7

MSC INTERNAL TECHNICAL NOTE

A MODIFICATION OF HERRICK'S SOLUTION OF THE TWO-BODY PROBLEM FOR ALL CASES

Prepared by: William H. Goodyear
Staff Mathematician
IBM FSD
RTCC Project

Prepared by: Lawrence F. Guseman, Jr.
Research Mathematician
Theory and Analysis Office

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

MANNED SPACECRAFT CENTER

HOUSTON, TEXAS

July 28, 1964

TABLE OF CONTENTS

SUMMARY

A solution of the two-body problem given by Herrick for many types of orbits is modified by using a different independent variable. The modification changes Herrick's transcendental functions and his form of Kepler's equation. The end result is a more general solution of the two-body problem in that it applies to both attractive and repulsive forces of any magnitude.

An outline is given for deriving the general modified solution from the equations for an elliptic orbit. This requires definition of transcendental functions which are then used to express Kepler's equation and give closed-form expressions for the series solution to the differential equations. A method is described in detail which outlines the computation necessary to determine coordinates at a given time from their known values at a given reference time. Formulas are also given for computing the partial derivatives of each of the resulting coordinates with respect to each of the reference coordinates.

LIST OF SYMBOLS

t	-	time
$x, y, z, \dot{x}, \dot{y}, \dot{z}$	-	rectangular position and velocity coordinates. When subscripted they refer to a particular point in time.
h	-	sum of the potential and kinetic energy
μ	-	gravitational constant
a	-	semi-major axis
E	-	eccentric anomaly
e	-	eccentricity
r	-	magnitude of position vector
$\dot{r}$	-	magnitude of change in position vector
v	-	magnitude of velocity vector
ψ	-	defined parameter used to generalize the equations for elliptic motion
$S_0, S_1, S_2, S_3, S_4, S_5$	-	defined transcendental functions
P	-	period of the orbit
f, g	-	power series in $(t-t_0)$
$\dot{f}, \dot{g}$	-	time derivatives of f and g.
Δt	-	residual used in solution of Kepler's equation

INTRODUCTION

A number of proven methods exist for obtaining a space trajectory from given initial conditions at some reference time (Encke's, Cowell's, Variation of Parameters, etc.). Most of these methods start with the undisturbed two-body space trajectory and consequently the need for a logically-simple, numerically-accurate solution of the two-body problem is apparent.

The usual approach to solving the two-body problem is to develop a separate method of solution for handling each of the different types of orbits encountered. The choice of the method to be used is made by logically testing numerical values of certain orbit-defining parameters which have been computed from a set of initial conditions.

In practice it is sometimes difficult to determine numerically the exact values of parameters for which one method is chosen as opposed to another (e.g., the problem of choosing the "best" when the energy is very small in absolute value, or zero). Consequently, it is advantageous to have a single method for solving the two-body problem which is continuous for all values of the orbit-defining parameters, thereby eliminating the logic associated with making a choice of methods.

A solution of the two-body problem is given by Herrick*[1] for all cases in which the constant μ in the differential equations

* When this work was completed, the authors were not aware of the work done by K. Stumpff [2].

$$\ddot{x} = -\mu x/r^3$$
$$\ddot{y} = -\mu y/r^3 \qquad r \equiv +\sqrt{x^2 + y^2 + z^2}$$
$$\ddot{z} = -\mu z/r^3 \tag{1}$$

is positive and relatively large. His solution may be modified to include all values of μ by utilizing the parameter

$$\psi = \frac{(E-E_0)}{\sqrt{\mu/a}} \tag{2}$$

rather than $\sqrt{a}(E-E_0)$ to generalize the equations for elliptic motion.

For an elliptic orbit, E is the eccentric anomaly for any time t at which the position coordinates are x, y, z, and the velocity coordinates are $\dot{x}, \dot{y}, \dot{z}$, and E_0 is the value of E for a particular reference time t_0 at which the position coordinates are x_0, y_0, z_0 and the velocity coordinates are $\dot{x}_0, \dot{y}_0, \dot{z}_0$. Also, a is the semi-major axis of the orbital ellipse, and

$$\mu/a = -2h \tag{3}$$

where the negative constant h is the sum of the kinetic and potential energy.

$$h = v^2/2 - \mu/r = v_0^2/2 - \mu/r_0 \tag{4}$$

Here the square of the magnitude of the velocity is

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \tag{5}$$

and r_0, v_0 are values of r, v at the time t_0.

The solution of the differential equations (1) expresses the relation between the coordinates x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$ at any time t and the coordinates x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ at a reference time t_0. In Section 1, the equations of this solution for elliptic orbits are generalized to include all types of orbits by using ψ rather than E or $(E-E_0)$ as the variable in Kepler's equation. This requires the expression of Herrick's transcendental functions in terms of ψ. In Section 2, a method is described for computing the coordinates x_1, y_1, z_1, $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$ at a given time t_1 from the known coordinates x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ at a given reference time t_0. In Section 3, formulas are given for computing the thirty-six partial derivatives of each of the coordinates x_1, y_1, z_1, $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$ with respect to each of the coordinates x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$.

1. DERIVATION OF GENERAL SOLUTION FROM ELLIPTIC CASE

1.1. Definition of the Transcendental Functions

The six transcendental functions S_0, S_1, S_2, S_3, S_4, S_5 are defined below in terms of parameters for elliptic motion, but are expressed in terms of $2h$ and ψ for the general solution

$$S_0 \equiv \cos(E-E_0) \tag{6a}$$

$$= 1 - (E-E_0)^2/2! + (E-E_0)^4/4! - (E-E_0)^6/6! + \ldots$$

$$= 1 + (-\tfrac{\mu}{a}\psi^2)/2! + (-\tfrac{\mu}{a}\psi^2)^2/4! + (-\tfrac{\mu}{a}\psi^2)^3/6! + \ldots$$

$$= 1 + (2h)^1\psi^2/2! + (2h)^2\psi^4/4! + (2h)^3\psi^6/6! + \ldots$$

$$S_1 \equiv \frac{\sin(E-E_0)}{\sqrt{\mu/a}} = \frac{(E-E_0)}{\sqrt{\mu/a}}\,\frac{\sin(E-E_0)}{(E-E_0)} \tag{6b}$$

$$= \psi[1 - (E-E_0)^2/3! + (E-E_0)^4/5! - (E-E_0)^6/7! + \ldots]$$

$$= \psi[1 + (-\tfrac{\mu}{a}\psi^2)/3! + (-\tfrac{\mu}{a}\psi^2)^2/5! + (-\tfrac{\mu}{a}\psi^2)^3/7! + \ldots]$$

$$= \psi^1/1! + (2h)^1\psi^3/3! + (2h)^2\psi^5/5! + (2h)^3\psi^7/7! + \ldots$$

$$S_2 \equiv \frac{1-\cos(E-E_0)}{(\sqrt{\mu/a})^2} = \frac{S_0 - 1}{2h} \tag{6c}$$

$$= \psi^2/2! + (2h)^1\psi^4/4! + (2h)^2\psi^6/6! + (2h)^3\psi^8/8! + \ldots$$

$$S_3 \equiv \frac{(E-E_0) - \sin(E-E_0)}{(\sqrt{\mu/a})^3} = \frac{S_1-\psi}{2h} \tag{6d}$$

$$= \psi^3/3! + (2h)^1\psi^5/5! + (2h)^2\psi^7/7! + (2h)^3\psi^9/9! + \ldots$$

$$S_4 \equiv \frac{(E-E_0)^2/2! - [1-\cos(E-E_0)]}{(\sqrt{\mu/a})^4} = \frac{S_2-\psi^2/2!}{2h} \tag{6e}$$

$$= \psi^4/4! + (2h)^1\psi^6/6! + (2h)^2\psi^8/8! + (2h)^3\psi^{10}/10! + \ldots$$

$$S_5 \equiv \frac{(E-E_0)^3/3! - [(E-E_0) - \sin(E-E_0)]}{(\sqrt{\mu/a})^5} = \frac{S_3-\psi^3/3!}{2h} \quad (6f)$$

$$= \psi^5/5! + (2h)^1\psi^7/7! + (2h)^2\psi^9/9! + (2h)^3\psi^{11}/11! + \ldots$$

These transcendental functions of ψ and h for the general solution replace the trigonometric functions of E or $(E-E_0)$ that apply only to elliptic motion.

1.2. Properties of the Transcendental Functions

The transcendental functions S_1, S_2, S_3 of h and ψ have the following important properties:

$$\frac{\partial S_1}{\partial \psi} = S \qquad \frac{\partial S_1}{\partial h} = \psi S_2 - 1.S_3$$

$$\frac{\partial S_2}{\partial \psi} = S_1 \quad (7a) \qquad \frac{\partial S_2}{\partial h} = \psi S_3 - 2.S_4 \quad (7b)$$

$$\frac{\partial S_3}{\partial \psi} = S_2 \qquad \frac{\partial S_3}{\partial h} = \psi S_4 - 3.S_5$$

The use of these relations is the reason for defining the functions S_0, S_1 and S_4, S_5 since the general solution of the differential equations (1) may be expressed in terms of S_2 and S_3 alone.

In the case of an elliptic orbit (for which h is negative) let

$$(\bar{t}-t_0) = (t-t_0) - mP \quad (8)$$

where

$$P = 2\pi/\sqrt{\mu/a^3} = 2\pi\mu/(\sqrt{-2h})^3 \quad (9)$$

is the period of the orbit, and m is a negative, zero, or positive integer which is chosen to minimize $|(\bar{t}-t_0)|$. Then

$$(\bar{E}-E_0) = (E-E_0) - 2\pi m \tag{10}$$

rather than $(E-E_0)$ will be related to $(\bar{t}-t_0)$, rather than $(t-t_0)$, by the elliptic form (13) of Kepler's equation. However, the coordinates $x, y, z, \dot{x}, \dot{y}, \dot{z}$ are the same for $(\bar{E}-E_0)$ and $(E-E_0)$ because of the periodicity of the orbit.

Similarly, in the special case of an elliptic orbit,

$$\bar{\psi} = \frac{\bar{E}-E_0}{\sqrt{\mu/a}} = \psi - m\frac{2\pi}{\sqrt{-2h}} \tag{11}$$

rather than ψ will be related to $(\bar{t}-t_0)$, rather than $(t-t_0)$, by the general form (16) of Kepler's equation. Also, the coordinates $x, y, z, \dot{x}, \dot{y}, \dot{z}$ are the same for $\bar{\psi}$ and ψ because of the periodicity of an elliptic orbit. The definitions (6a) - (6f), (10), and (11) can then be used to show that

$$\begin{aligned}
S_0 &= \bar{S}_0 \\
S_1 &= \bar{S}_1 \\
S_2 &= \bar{S}_2 \\
S_3 &= \bar{S}_3 + m[2\pi/(\sqrt{-2h})^3] \\
S_4 &= \bar{S}_4 + m[2\pi/(\sqrt{-2h})^3](\psi+\bar{\psi})/2 \\
S_5 &= \bar{S}_5 + m[2\pi/(\sqrt{-2h})^3][\bar{\psi}^2+\bar{\psi}\psi+\psi^2)/6 + 1/2h],
\end{aligned} \tag{12}$$

where $\bar{S}_0, \bar{S}_1, \bar{S}_2, \bar{S}_3, \bar{S}_4, \bar{S}_5$ are respectively the transcendental functions (6a) through (6f) of $\bar{\psi}$ and h rather than ψ and $\dot{h}$.

1.3. Kepler's Equation and Its Derivatives

Kepler's equation for an elliptic orbit may be expressed in the form

$$
\begin{aligned}
(t-t_0) &= \frac{M-M_0}{\sqrt{\mu/a^3}} \\
&\equiv \frac{(E-e\sin E) - (E_0 - e\sin E_0)}{\sqrt{\mu/a^3}} \qquad (13) \\
&= a(1-e\cos E_0)\frac{(E-E_0)}{\sqrt{\mu/a}} + \sqrt{\mu a}\,e \sin E_0 \frac{1-\cos(E-E_0)}{(\sqrt{\mu/a})^2} \\
&\quad + \mu e\cos E_0 \frac{(E-E_0)-\sin(E-E_0)}{(\sqrt{\mu/a})^3}
\end{aligned}
$$

In these equations, e is the eccentricity of the orbital ellipse. One expression for $(M-M_0)/\sqrt{\mu/a^3}$ is in terms of E and the other expression is in terms of $(E-E_0)$. These two expressions are trigonometric identities, but the second is more general in that it is valid for circular and near-circular orbits.

The parameters r, $(r\dot{r})$ and $(rv^2 - \mu)$ for an elliptic orbit may be expressed in the forms

$$
\begin{aligned}
r &\equiv {}^{+}\sqrt{x^2+y^2+z^2} \qquad (14a) \\
&= a(1-e\cos E)
\end{aligned}
$$

$$= a(1 - e\cos E_0) + \sqrt{\mu a}\, e\sin E_0 \frac{\sin(E-E_0)}{\sqrt{\mu/a}}$$

$$+ \mu e \cos E_0 \frac{1-\cos(E-E_0)}{(\sqrt{\mu/a})^2}$$

$$(r\dot{r}) \equiv x\dot{x} + y\dot{y} + z\dot{z} \tag{14b}$$

$$= \sqrt{\mu a}\, e\sin E$$

$$= \sqrt{\mu a}\, e \sin E_0 \cos(E-E_0) + \mu e \cos E_0 \frac{\sin(E-E_0)}{\sqrt{\mu/a}}$$

$$(rv^2 - \mu) \equiv {}^{+}\sqrt{x^2 + y^2 + z^2}\,(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \mu \tag{14c}$$

$$= \mu e \cos E_0$$

$$= \mu e \cos E_0 \cos(E-E_0) + (-\frac{\mu}{a})\sqrt{\mu a}\, e\sin E \frac{\sin(E-E_0)}{\sqrt{\mu/a}}$$

Equation (14a) is r times the time derivative of Kepler's equation (13). Similarly, (14b) is r times the time derivative of (14a), and (14c) is r times the time derivative of (14b). For the particular case where t is t_0 and E is thus E_0, equations (14a) - (14c) become

$$r_0 = a(1-e \cos E_0) \tag{15a}$$

$$(r_0\dot{r}_0) = \sqrt{\mu a}\, e \sin E_0 \tag{15b}$$

$$(r_0 v_0^2 - \mu) = \mu e \cos E_0 \tag{15c}$$

The general form of Kepler's equation and the general expressions for r, $(r\dot{r})$ and $(rv^2-\mu)$ are obtained by substituting:

r_0, $(r_0\dot{r}_0)$, $(r_0v^2-\mu)$ given by (15a) - (15c),

ψ defined by (2),

S_0, S_1, S_2, S_3 given by (6a) - (6d) and (2h) defined by (3), into equations (13) and (14a) - (14c).

Kepler's equation for the general case is thus

$$(t-t_0) = r_0\psi + (r_0\dot{r}_0)\, S_2 + (r_0 v_0^2 - \mu)S_3 \qquad (16)$$

in terms of ψ and its transcendental functions S_2 and S_3. Also, r, $(r\dot{r})$, and $(rv^2 - \mu)$ are

$$r = r_0 + (r_0\dot{r}_0)S_1 + (r_0 v_0^2 - \mu)S_2 \qquad (17a)$$

$$(r\dot{r}) = (r_0\dot{r}_0)S_0 + (r_0 v_0^2 - \mu)S_1 \qquad (17b)$$

$$(rv^2 - \mu) = (r_0\dot{r}_0)(2h)S_1 + (r_0 v_0^2 - \mu)S_0 \qquad (17c)$$

in terms of the transcendental functions S_0, S_1, S_2, S_3. Use of the properties of (7a) shows that r, $r\dot{r}$, and $(rv^2 - \mu)$ are the successive derivatives of $(t-t_0)$ with respect to ψ. This fact facilitates the graphical interpretation of Kepler's equation (16) in which $(t-t_0)$ is plotted as a function of ψ for all the various types of orbits. These include the circular, elliptic, parabolic, hyperbolic, and rectilinear cases for an attractive force, as well as the hyperbolic and rectilinear cases for a repulsive force.

1.4. Closed Form Expressions for the Series Solution

The series solution to the differential equations (1) is usually expressed in the form

$$[x,\ y,\ z] = f\cdot[x_0,\ y_0,\ z_0] + g\cdot[\dot{x}_0,\ \dot{y}_0,\ \dot{z}_0] \qquad (18a)$$

$$[\dot{x},\ \dot{y},\ \dot{z}] = \dot{f}\cdot[x_0,\ y_0,\ z_0] + \dot{g}\cdot[\dot{x}_0,\ \dot{y}_0,\ \dot{z}_0] \qquad (18b)$$

where f, g and their time derivatives $\dot{f}$, $\dot{g}$ are infinite power series in $(t-t_0)$ whose successive coefficients are increasingly complicated functions of x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$. However, the functions f, g, $\dot{f}$, $\dot{g}$ can be expressed in closed form, in terms of the parameter $(E-E_0)$, for an elliptic orbit.

$$f = 1 - \frac{1-\cos(E-E_0)}{r_0/a} \qquad (19a)$$

$$g = (t-t_0) - \frac{(E-E_0) - \sin(E-E_0)}{\sqrt{\mu/a^3}} \qquad 919b)$$

$$\dot{f} = -\sqrt{\frac{\mu}{a^3}}\frac{\sin(E-E_0)}{(r_0/a)(r/a)} \qquad (19c)$$

$$\dot{g} = 1 - \frac{1-\cos(E-E_0)}{r/a} \qquad (19d)$$

These formulas are easily expressed in terms of S_1, S_2, S_3 by use of the definitions (6b), (6c), (6d). The closed-form expressions for f, g, $\dot{f}$, $\dot{g}$ in the general case are thus

$$f = 1 - \mu S_2/r_0$$

$$g = (t-t_0) - \mu S_3 \qquad (20)$$

$$\dot{f} = -\mu S_1/(r r_0)$$

$$\dot{g} = 1 - \mu S_2/r$$

in terms of the transcendental functions S_1, S_2, S_3 and the parameters r, r_0, μ and $(t-t_0)$.

2. A METHOD FOR COMPUTATION OF COORDINATES

2.1. Initial Computations and Start of Iterations

The first step in computing the numerical values of the coordinates x_1, y_1, z_1, $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$ at a given time t_1 from the given coordinates x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ at the given time t_0 is to calculate the parameters

$$r_0 = {}^{+}\sqrt{x_0{}^2 + y_0{}^2 + z_0{}^2} \tag{21a}$$

$$(r_0\dot{r}_0) = x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0 \tag{21b}$$

$$(v_0^2) = \dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2 \tag{21c}$$

$$(2h) = v_0^2 - 2\mu/r_0 \tag{21d}$$

$$(r_0 v_0^2 - \mu) = r_0\ (v_0^2) - \mu \tag{21e}$$

$$(t_1 - t_0) = t_1 - t_0 \tag{21f}$$

from the coordinates x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$, the constant μ , and the times t_0 and t_1 .

When (2h) obtained in (21d) is negative and the orbit is thus periodic, the period

$$P = 2\pi\mu/(\sqrt{-2h})^3 \tag{22}$$

of the orbit is calculated and

$$m = \text{INTEGER portion of } [(t_1 - t_0)/p + 1/2] \tag{23}$$

is determined. This minimizes the absolute value of

$$(\bar{t}_1 - t_0) = (t_1 - t_0) - mp \tag{24}$$

which is calculated and used in the place of (t_1-t_0) to determine $x_1, y_1, z_1, \dot{x}_1, \dot{y}_1, \dot{z}_1$ in all the computations that follow. Because of this, no distinction is made between $(\bar{t}_1-t_0)$ and (t_1-t_0) or quantities computed from either in formulas given below to determine $x_1, y_1, z_1, \dot{x}_1, \dot{y}_1, \dot{z}_1$.

Kepler's equation (16) must be solved by an iterative scheme to determine the value ψ_1 for ψ which corresponds to (t_1-t_0). That is, that value ψ_1 of ψ must be found which makes the right hand side of Kepler's equation (16) equal to (t_1-t_0). In describing these computations below, ψ will be used to represent a current approximation for ψ_1, and ψ' will be used to represent a further approximation calculated from the approximation ψ. The initial value for ψ is

$$\psi = (t_1-t_0)/r_0 \tag{25}$$

which is computed and then used to evaluate the transcendental functions as described in 2.2.

2.2. Evaluation of the Transcendental Functions

The transcendental functions S_4 and S_5 are first computed from the current approximation ψ for ψ_1, by using equations (6e) and (6f) in the forms

$$S_4 = \psi^4[1/4!+(2h\psi^2)/6! + (2h\psi^2)^2/8! + (2h\psi^2)^3/10! + \ldots]$$
$$S_5 = \psi^5[1/5!+(2h\psi^2)/7! + (2h\psi^2)^2/9! + \ldots] \qquad (26)$$

The accurate computation of each of the series in brackets in these two equations is an important numerical problem. A simple solution is to forward sum each of the series term by term until the addition of another term does not change either sum. Then the accuracy of the summations may be improved if desired by backward nesting the same number of terms used in the forward summations. Multiplication of the two sums by ψ^4 and ψ^5 respectively then gives S_4 and S_5.

The functions S_3, S_2, S_1, S_0 are then computed by using the relations

$$\begin{aligned} S_3 &= \psi^3/6 + (2h)S_5 \\ S_2 &= \psi^2/2 + (2h)S_4 \\ S_1 &= \psi + (2h)S_3 \\ S_0 &= 1 + (2h)S_2 \end{aligned} \qquad (27)$$

which are obtained from equations (6f) back through (6c). The functions S_2 and S_3 could be computed directly by two equations similar to (26) above, and could then be used to compute x_1, y_1, z_1, $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$. However, S_4 and S_5 cannot in general be accurately computed from S_2 and S_3 , and S_4 and S_5 are required if partial derivatives are desired. The

functions S_0 and S_1 are defined and computed merely for convenience of notation.

2.3. The Solution of Kepler's Equation

The value $(t-t_0)$, corresponding to the value ψ and its functions S_2, S_3, is first computed by

$$(t-t_0) = r_0\psi + (r_0\dot{r}_0)S_2 + (r_0v_0^2 - \mu)S_3 \tag{28}$$

which is Kepler's equation (16). That is, if $(t-t_0)$ were the time interval at which a solution were desired, ψ would be the solution of Kepler's equation. However, the iterative procedure must find that particular ψ for which the residual

$$\Delta t = (t-t_0) - (t_1-t_0) \tag{29}$$

is zero. This particular value of ψ will then be the correct value for ψ_1 . The residual (29) for the current value of ψ is computed along with the current value of r which corresponds to ψ and its functions S_1, S_2.

$$r = r_0 + (r_0\dot{r}_0)S_1 + (r_0v_0^2 - \mu)S_2 \tag{30}$$

This r, is also the derivative $d(\Delta t)/d\psi$ and is therefore the slope of the curve of $(t-t_0)$ as a function of ψ .

Newton's method is now applied to determine a new approximation ψ' for ψ_1 .

$$\psi' = \psi - \Delta t/r \tag{31}$$

Then the transcendental functions S'_4, S'_5 and S'_3 , S'_2, S'_1, S'_0 of ψ' are computed by applying the formulas in Section 2.2. but using ψ' rather than ψ . Also, the results are used in equations (29), (30), and (31) above to obtain values $(t'-t_0)$, $\Delta t'$, and r' which correspond to ψ' and its functions S'_1, S'_2 S'_3 . If the residual $\Delta t'$ is then less in absolute magnitude than Δt, then ψ', the transcendental functions S'_0, S'_1, S'_2. S'_3, S'_4, S'_5 and the functions $(t'-t_0)$, $\Delta t'$, r' are all accepted as new values for ψ and S_0, S_1, S_2, S_3, S_4, S_5 and $(t-t_0)$, Δt, r . Then Newton's method (31) is used to compute a new ψ' and repeat the entire computation.

When $\Delta t'$ is not less in absolute magnitude than Δt, the current ψ' is not accepted as a new value for ψ . Rather, a different value for ψ' is computed by setting n equal to unity in the equation

$$\psi' = \psi - \Delta t/n \tag{32}$$

Then the slope r has been replaced by unit slope to determine the new approximation ψ' for ψ_1.

This ψ' is then used to compute its functions S_0', S_1', S_2', S_3', S_4' and $(t'-t_0)$, $\Delta t'$, r'. If $\Delta t'$ is then less in absolute magnitude than Δt, ψ' and its functions are accepted as new values for ψ and its functions, and Newton's method (31) is again applied as described above.

If $\Delta t'$ is not less in absolute magnitude than Δt, n in equation (32) is doubled to compute a different ψ' . The slope n is repeatedly doubled until a $\Delta t'$ is obtained which is less than Δt in absolute magnitude, or until ψ and ψ' are numerically identical. When the latter is true, the resulting ψ is accepted as the value of ψ_1 for computing the coordinates x_1, y_1, z_1, $\dot{x}_1$, $\dot{y}_1$; $\dot{z}_1$.

2.4. Computation of Coordinates

Since ψ is now the correct value for ψ_1, the functions f, g, $\dot{f}$, $\dot{g}$ in equations (20) are the functions for the coordinates x_1, y_1, z_1, $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$. Therefore, these functions are computed from (t_1-t_0) and the functions S_1, S_2, S_3 and r_1 of ψ_1 .

$$\begin{aligned} f &= 1 - \mu S_2/r_0 \\ g &= (t_1-t_0) - \mu S_3 \\ \dot{f} &= -\mu S_1/(r_1 r_0) \\ \dot{g} &= 1 - \mu S_2/r_1 \end{aligned} \qquad (33)$$

These functions are then used to compute x_1, y_1, z_1, $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$ by using equations (18a) and (18b).

$$
\begin{aligned}
x_1 &= f x_0 + g \dot{x}_0 \\
y_1 &= f y_0 + g \dot{y}_0 \\
z_1 &= f z_0 + g \dot{z}_0 \\
\dot{x}_1 &= \dot{f} x_0 + \dot{g} \dot{x}_0 \\
\dot{y}_1 &= \dot{f} y_0 + \dot{g} \dot{y}_0 \\
\dot{z}_1 &= \dot{f} z_0 + \dot{g} \dot{z}_0
\end{aligned}
\qquad (34)
$$

Thus the coordinates x_1, y_1, z_1, $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$ at time t_1 have been computed from the constant μ, the time t_0 and the coordinates x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ at time t_0.

3. A METHOD FOR COMPUTING PARTIAL DERIVATIVES

3.1. Outline of Derivation of Partial Derivatives

Let the times t_0 and t_1 as well as μ be treated as fixed constants and let the coordinates x_1, y_1, z_1, $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$ be treated as dependent variables of x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ which are treated as independent variables. The thirty-six partial derivatives of each of the coordinates x_1, y_1, z_1, $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$ with respect to each of the coordinates x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ have many important practical applications. These derivatives are obtained by chain differentiation of the relations in Section 2 that are used to compute x_1, y_1, z_1, $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$ from x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$. The chain differentiations must then be combined to obtain tractable formulas for computing the partial derivatives.

The chain differentiations are rather tedious and lengthy and will therefore not be given here. The whole procedure is facilited by the use of matrix notation. The basic idea is to obtain matrix relationships between all the differentials of quantities which are direct or indirect functions of x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$. These results are then combined to eliminate all differentials other than dx_1, dy_1, dz_1, $d\dot{x}_1$, $d\dot{y}_1$, $d\dot{z}_1$ and dx_0, dy_0, dz_0, $d\dot{x}_0$, $d\dot{y}_0$, $d\dot{z}_0$. Then

the coefficient matrix relating these differentials is the desired matrix of the thirty-six partial derivatives.

3.2. Evaluation of Parameters and Periodicity Computations

The parameter r_1 and the transcendental functions S_0, S_1, S_2, S_3, S_4, S_5 of ψ_1 have been determined in the computations described in Section 2 to obtain the coordinates x_1, y_1, z_1, $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$. The parameter $\dot{r}_1$ must also be computed from

$$\dot{r}_1 = [(r_0\dot{r}_0)S_0 + (r_0 v_0^2 - \mu)S_1]/r_1 \qquad (35)$$

and $\frac{\partial S_1}{\partial h}$, $\frac{\partial S_2}{\partial h}$, $\frac{\partial S_3}{\partial h}$ must be computed from (7b)

$$\begin{aligned} \frac{\partial S_1}{\partial h} &= \psi_1 S_2 - S_3 \\ \frac{\partial S_2}{\partial h} &= \psi_1 S_3 - 2S_4 \\ \frac{\partial S_3}{\partial h} &= \psi_1 S_4 - 3S_5 \end{aligned} \qquad (36)$$

In addition, the true values of ψ_1, S_3, S_4, S_5 must be determined from $\bar{\psi}_1$, $\bar{S}_3$, $\bar{S}_4$, $\bar{S}_5$ by using the equations of (12) if the orbit is elliptic and m is not zero. In the computations of Section 2, no distinction was made between $\bar{\psi}_1$, $\bar{S}_3$, $\bar{S}_4$, $\bar{S}_5$ computed from $(\bar{t}_1 - t_0)$ and the true values ψ_1, S_3, S_4, S_5 which are the same functions of $(t_1 - t_0)$.

However, this distinction must be made in order to obtain the correct partial derivatives from the formulas in Sections 3.3. and 3.4.

$$
\begin{aligned}
\psi &= \bar{\psi} + m\,(2\pi/\sqrt{-2h}) \\
S_3 &= \bar{S}_3 + m[2\pi/(\sqrt{-2h})^3] \\
S_4 &= \bar{S}_4 + m[2\pi/(\sqrt{-2h})^3](\psi + \bar{\psi})/2 \\
S_5 &= \bar{S}_5 + m[2\pi/(\sqrt{-2h})^3][(\bar{\psi}^2+\bar{\psi}\psi+\psi^2)/6 + 1/2h]
\end{aligned}
\tag{37}
$$

The functions S_0, S_1, S_2 need not be recomputed since they are equal to $\bar{S}_0$, $\bar{S}_1$, $\bar{S}_2$ by (12).

3.3. Evaluation of the Four by Four Matrix

The four by four matrix below is first calculated as an intermediate step for the computation of the partial derivatives. The letter T is used to indicate the matrix transpose of the two column matrices.

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
= \frac{\mu}{r_1}
\begin{bmatrix}
S_1 \\
S_2 \\
(S_0 - \dot{r}_1 S_1)/r_1 \\
(S_1 - \dot{r}_1 S_2)/r_1
\end{bmatrix}
\begin{bmatrix}
\psi_1 + v_0^2 S_3 + \frac{\mu}{r_0^2}\left[(r_0\dot{r}_0)\frac{\partial S_2}{\partial h} + (r_0 v_0^2 - \mu)\frac{\partial S_3}{\partial h}\right] \\
S_2 \\
r_0 S_2 \\
2r_0 S_3 + \left[(r_0\dot{r}_0)\frac{\partial S_2}{\partial h} + (r_0 v_0^2 - \mu)\frac{\partial S_3}{\partial h}\right]
\end{bmatrix}^T
$$

$$
+\frac{\mu}{r_1}
\begin{bmatrix}
0 \\
0 \\
S_1/r_1 \\
S_2/r_2
\end{bmatrix}
\begin{bmatrix}
1 + v_0^2 S_2 + \frac{\mu}{r_0^2}\left[(r_0\dot{r}_0)\frac{\partial S_1}{\partial h} + (r_0 v_0^2 - \mu)\frac{\partial S_2}{\partial h}\right] \\
S_1 \\
r_0 S_1 \\
2r_0 S_2 + \left[(r_0\dot{r}_0)\frac{\partial S_1}{\partial h} + (r_0 v_0^2 - \mu)\frac{\partial S_2}{\partial h}\right]
\end{bmatrix}^T
$$

$$
- \begin{bmatrix}
\frac{\mu}{r_0}\left(\frac{\mu}{r_0}\frac{\partial S_2}{\partial h} - S_2\right) & 0 & 0 & \mu\frac{\partial S_2}{\partial h} \\
\left(\frac{\mu}{r_0}\right)^2 \frac{\partial S_3}{\partial h} & 0 & 0 & \mu\frac{\partial S_3}{\partial h} \\
\frac{\mu/r_0}{r_1}\left(\frac{\mu}{r_0}\frac{\partial S_1}{\partial h} - S_1\right) & 0 & 0 & \frac{\mu}{r_1}\frac{\partial S_1}{\partial h} \\
\frac{(\mu/r_0)^2}{r_1}\frac{\partial S_2}{\partial h} & 0 & 0 & \frac{\mu}{r_1}\frac{\partial S_2}{\partial h}
\end{bmatrix}
$$

3.4. Computation of the Partial Derivatives

The two by two sub-matrices of the four by four matrix above are then used to calculate the partial derivatives by the formulas given below. The letter T indicates the matrix transpose of the three by two matrices.

$$
\begin{bmatrix}
\frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\
\frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\
\frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0}
\end{bmatrix}
=
\begin{bmatrix}
f & 0 & 0 \\
0 & f & 0 \\
0 & 0 & f
\end{bmatrix}
+
\begin{bmatrix}
\frac{x_0}{r_0} & \dot{x}_0 \\
\frac{y_0}{r_0} & \dot{y}_0 \\
\frac{z_0}{r_0} & \dot{z}_0
\end{bmatrix}
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
\frac{x_0}{r_0} & \dot{x}_0 \\
\frac{y_0}{r_0} & \dot{y}_0 \\
\frac{z_0}{r_0} & \dot{z}_0
\end{bmatrix}^T
$$

$$\begin{bmatrix} \frac{\partial x}{\partial \dot{x}_0} & \frac{\partial x}{\partial \dot{y}_0} & \frac{\partial x}{\partial \dot{z}_0} \\ \frac{\partial y}{\partial \dot{x}_0} & \frac{\partial y}{\partial \dot{y}_0} & \frac{\partial y}{\partial \dot{z}_0} \\ \frac{\partial z}{\partial \dot{x}_0} & \frac{\partial z}{\partial \dot{y}_0} & \frac{\partial z}{\partial \dot{z}_0} \end{bmatrix} = \begin{bmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{bmatrix} + \begin{bmatrix} \frac{x_0}{r_0} & \dot{x}_0 \\ \frac{y_0}{r_0} & \dot{y}_0 \\ \frac{z_0}{r_0} & \dot{z}_0 \end{bmatrix} \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} \frac{x_0}{r_0} & \dot{x}_0 \\ \frac{y_0}{r_0} & \dot{y}_0 \\ \frac{z_0}{r_0} & \dot{z}_0 \end{bmatrix}^T$$

$$\begin{bmatrix} \frac{\partial \dot{x}}{\partial x_0} & \frac{\partial \dot{x}}{\partial y_0} & \frac{\partial \dot{x}}{\partial z_0} \\ \frac{\partial \dot{y}}{\partial x_0} & \frac{\partial \dot{y}}{\partial y_0} & \frac{\partial \dot{y}}{\partial z_0} \\ \frac{\partial \dot{z}}{\partial x_0} & \frac{\partial \dot{z}}{\partial y_0} & \frac{\partial \dot{z}}{\partial z_0} \end{bmatrix} = \begin{bmatrix} \dot{f} & 0 & 0 \\ 0 & \dot{f} & 0 \\ 0 & 0 & \dot{f} \end{bmatrix} + \begin{bmatrix} \frac{x_0}{r_0} & \dot{x}_0 \\ \frac{y_0}{r_0} & \dot{y}_0 \\ \frac{z_0}{r_0} & \dot{z}_0 \end{bmatrix} \begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \begin{bmatrix} \frac{x_0}{r_0} & \dot{x}_0 \\ \frac{y_0}{r_0} & \dot{y}_0 \\ \frac{z_0}{r_0} & \dot{z}_0 \end{bmatrix}^T$$

$$\begin{bmatrix} \frac{\partial \dot{x}}{\partial \dot{x}_0} & \frac{\partial \dot{x}}{\partial \dot{y}_0} & \frac{\partial \dot{x}}{\partial \dot{z}_0} \\ \frac{\partial \dot{y}}{\partial \dot{x}_0} & \frac{\partial \dot{y}}{\partial \dot{y}_0} & \frac{\partial \dot{y}}{\partial \dot{z}_0} \\ \frac{\partial \dot{z}}{\partial \dot{x}_0} & \frac{\partial \dot{z}}{\partial \dot{y}_0} & \frac{\partial \dot{z}}{\partial \dot{z}_0} \end{bmatrix} = \begin{bmatrix} \dot{g} & 0 & 0 \\ 0 & \dot{g} & 0 \\ 0 & 0 & \dot{g} \end{bmatrix} + \begin{bmatrix} \frac{x_0}{r_0} & \dot{x}_0 \\ \frac{y_0}{r_0} & \dot{y}_0 \\ \frac{z_0}{r_0} & \dot{z}_0 \end{bmatrix} \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \frac{x_0}{r_0} & \dot{x}_0 \\ \frac{y_0}{r_0} & \dot{y}_0 \\ \frac{z_0}{r_0} & \dot{z}_0 \end{bmatrix}^T$$

CONCLUDING REMARKS

The authors have written a FORTRAN IV program (available upon request) for computing coordinates and partial derivatives of the two-body problem. Cases run to test the program included: (for $\mu>0$) elliptic circular, parabolic, hyperbolic, rectilinear; and (for $\mu\leq 0$) hyperbolic, rectilinear.

Computationally, the program is superior to available programs in that it produces solutions and partial derivatives for all cases of the two-body problem without exception. It also has no disadvantage in the accuracy and speed of computation.

References

1. Herrick, Samuel: Astrodynamical Report Number 7 - Positions, Velocities, Ephemerides Referred to the Dynamic Center, Air Research and Development Command/Air Force Office of Scientific Research, Mechanics Division/Contract Number AF49 (638)-498, ASTIA Document Number AD 250757. Paragraph 6 Q.

2. Stumpff, Karl: Himmelsmechanik, Band I, VEB Verlag, Berlin, 1959, Chapter 5.